Preface and Outline

The era with satellites has redrawn our world, by measuring it so accurately. Professions are changed (and even created). Satellite based positioning is a new technology that already influences the lives of many people. So many good ideas go into the success (and the amazing accuracy) of global positioning. Our goal is to present those ideas in two ways:

1. **By clear exposition** (so you will understand them)

2. **By short MATLAB codes** (so you can use them).

The ideas bring together multiple technologies. That is the secret of global positioning, to combine engineering tools with computational algorithms (and the mathematics of coordinate mappings). Most of us are more competent and interested in particular parts of this whole construction; those sections of the book will be read more closely.

This book describes the linear algebra and the specialized algorithms for receivers to compute their positions. The collective name *Global Navigation Satellite Systems* (GNSS) includes satellite based augmentation to help the user with real-time ionospheric corrections. The augmentation systems include WAAS in USA, EGNOS in Europe, GAGAN in India, and QZSS in Japan. When we come to the algorithm level, using pseudoranges and carrier ranges and corrections, we only experience small differences in these systems. Our emphasis is on GPS as this is the most used system, and we offer details on EGNOS too. Our examples and data sets will come from The GPS Center in Aalborg.

Chapter 1 already shows the variety of tools that come together into accurate positioning. Then the book explains how that information is used.

1.1 We start with an informal survey of applications. Lower accuracy comes from pseudoranges alone, high accuracy uses also carrier phases.

1.2 The central computations find pseudoranges and their corrections. The underlying algorithms (in Chapters 9 and 10) are introduced in *A GPS Software Receiver*.

1.3 We present the characteristics of different GNSS systems, existing and planned.
Positioning systems continue to improve because more satellite signals are available, with added information and increasing accuracy. The many planned satellite launches witness how early in the process we are.

Chapters 9 and 10 explain the positioning algorithms that are at the heart of this subject. The ideas are straightforward, but the details (with removal of any error we can estimate) are more involved. We could not expect you to write codes for every step. The 18 MATLAB codes collected as “Easy Suites” are a vitally important part of this book.

Those Easy codes are listed on page 260. The receiver position (corrected for the all-important error in the clock) is computed from one-way pseudoranges. Normally more than four satellites are in view. Then we use least squares to estimate our position and our clock error. Other errors remain, whose sources we begin to understand.

Each satellite sends signals on more than one frequency. Chapter 10 shows how a more precise receiver can use phase observations for high accuracy. Signals are delayed in the ionosphere (light travels more slowly), and with two frequencies we can remove this error. A key idea in Chapter 10 is to work with differences in the observations at nearby receivers. This chapter ends with real-time kinematic positioning, including codes.

All those steps are possible because signals from the satellites contain so much information. Chapter 2 explains the pseudorandom sequences and Gold codes used by these signals. We also give details of modulations for readers who are involved with this aspect of the complete GPS technology.

Chapter 3 focuses on coordinates and changes of coordinates. Unfortunately the Earth is not a sphere. It is flattened at the poles by a factor close to 1/297. The first approximation to the geoid is an ellipsoid of revolution (an ellipse rotated around a North-South axis). GPS needs to convert latitude and longitude on the Earth to and from x-y-z Cartesian coordinates in space. The World Geodetic System (WGS 84) provides accurate parameters for positioning.

Chapters 4 to 8 describe the mathematics we need. There are two major ideas that appear everywhere in scientific computing:

1. Noise in the signal is modeled as a random process
2. The best estimate of a position is based on weighted least squares.

The crucial link between 1 and 2 is that the optimal weights in least squares are the inverses of the covariance matrices for the random variables. These variances and covariances must be estimated. This is never easy. The covariances go into the least-squares algorithms. The output must include an estimate (of position or velocity) and also a measure of confidence in that estimate.

The Kalman filter was created to produce this output: estimate plus covariance. The algorithm is recursive least squares, in which the output at time epoch $k$ is combined with new observations to produce an estimate at epoch $k+1$ (with covariance). Recursive means: The observations prior to $k$ are not used again, their contribution is made in estimating the state at time $k$. The update multiplies by the gain matrix.

A second feature of the Kalman filter is equally valuable. It can estimate a state that is changing. The receiver can move, and that movement is given by the state equation. The
update has to account for change in the state as well as new observations. The steps are conceptually simple but the algebra is notoriously painful (in comparison with the basic normal equation $A^T A x = A^T b$ for the unweighted least-squares solution of $A x = b$).

We do our best with that algebra. Chapter 8 in this book is modeled on Chapter 17 in our previous book, Linear Algebra, Geodesy, and GPS, which many readers found to be a clear and self-contained reference to the Kalman filter. And the implementation must be numerically stable, which is optimized by computing with orthogonal matrices wherever possible. Rank deficiency can occur, and needs special care.

Chapters 11 and 12 return to fundamental questions of geometry. We are computing on an ellipsoid. The geodesics are no longer great circles on a sphere—nothing so simple. A whole succession of famous geometers has worked with formulas on the ellipsoid and conformal mappings onto the plane. In presenting these fundamental issues of geodesy (Earth measurement), we are reporting on the achievements of the past. In a wonderful way, the new technology of satellite positioning has brought incredible accuracy to this classical science. The mathematics was always accurate, at last we have observations to match the formulas.

This is an inspiring subject and an exciting time. Archimedes and Gauss would be fascinated to see it now. We all hope they would be proud of us.

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